Prospective teachers' development of geometric reasoning through an exploratory approach

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We aim to characterize how prospective teachers perform in defining and classifying quadrilaterals through working on exploratory tasks. Data was gathered from the participants' reports and portfolios. Results show most understood the meaning of defining and presented correct definitions, using properties they previously ignored, and showing comprehension of the underlying concepts. They produced economical definitions in few cases, and performed better in inductive than deductive reasoning. The classifications showed conflicts between prior classifications and structural criteria that rules a geometrical classification. The exploratory work allowed participants to construct their knowledge in a meaningful way and reflection played an important role in becoming aware of personal preconceptions and knowledge.

Keywords: Teacher Education, Geometry, Geometric reasoning, Exploratory approach.

INTRODUCTION

This paper addresses prospective elementary teachers' preparation in geometry. Recent studies in Portugal show less than satisfactory results concerning the geometric knowledge they present before and after attending their teacher education programs (Menezes, Serrazina & Fonseca, 2014; Tempera, 2010). A similar conclusion is also found in studies from other countries, concerning teachers and prospective teachers, indicating that geometry is an area in which they perform poorly, have little self-confidence, and show weak geometric vocabulary (Clements & Sarama, 2011; Fujita & Jones, 2006; Jones, Mooney & Harries, 2002). In addition, there are very different views about what geometry can or should be taught in teacher preparation courses, which is problematic as the success of the teachers' work depends, to a great extent, on their deep understanding of geometry. And, we must also remember that knowing geometry does not ensure effectiveness, how teachers come to know it matters as well (Jones, Mooney & Harries, 2002).

This scenario challenges us to improve teacher's curriculum and preparation in this area and investigate its outcomes. The work we report in this paper fits into a wider study with that goal. We developed a design research experiment in the context of a curricular unit of geometry based on exploratory work, linking geometry and didactics and valuing prospective teachers' reflection on their learning. We seek to characterize how prospective teachers perform in processes which are components of geometric reasoning, focusing on defining, but also looking at classifying.

CONCEPTUAL FRAMEWORK

Prospective elementary teacher education in geometry

For the National Council of Teachers of Mathematics [NCTM] the knowledge necessary for teaching includes

> the content and discourse of mathematics, including mathematical concepts and procedures and the connections among them; multiple representations of mathematical concepts and procedures; ways to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality. (1991, p. 132).

This perspective is coherent with the idea advocated by Ma (1999) that teachers need a profound understanding of fundamental mathematics. But what does this mean in geometry? The NCTM (1991) states that all teachers should understand how geometry is used to describe the world we leave in and how it is used to solve concrete problems; analyze a diverse set of two and three dimensional figures; use synthetic geometry, coordinates and transformations; improve their skills in producing arguments, justifications and privilege spatial visualization. In 2000, the Conference Board for the Mathematical Sciences (CBMS) proposed that prospective K-5 teachers must develop competence in the following areas: Visualization skills (projections, cross-sections, decompositions; representing 3D objects in 2D and constructing 3D objects from 2D representations); basic shapes, their properties, and relationships among them (angles, transformations, congruence and similarity); and communicating geometric ideas (learning technical vocabulary and understanding the role of mathematical definition). The recent report of CBMS (2012) updates the main ideas for teaching preparation in geometry, presenting less topics and less complex competencies:

- Understanding geometric concepts of angle, parallel, and perpendicular, and using them in describing and defining shapes; describing and reasoning about spatial locations (including the coordinate plane).
- Classifying shapes into categories and reasoning to explain relationships among the categories.
- Reason about proportional relationships in scaling shapes up and down. (p. 30)

This shift confirms the lack of agreement about the geometric knowledge teachers should hold. In addition, the education of teachers concern also the ways they are taught. Regarding the results of several studies about prospective teachers' knowledge of mathematics, Watson and Mason (2007) propose that courses should prompt participants to engage in mathematical thinking through working on suitable mathematics tasks, develop their understanding about the features and power of those tasks, reflect on the experience of doing mathematics tasks individually or with others, challenge approaches dominated by procedures which depend on rote memorization and observe and listen to learners. These orientations are also consistent with ideas underlined by other investigators: in teacher education, the prospective teachers should learn using the same methods that are recommended they should use in the future (Ponte & Chapman, 2008); connecting subject matter knowledge and pedagogy is a promising strategy to develop both kinds of knowledge and their integration, which is critical to teach well (Ball, 2000). The work we conducted follows these proposals, as we focus on prospective teachers' learning as they work on exploratory tasks and reflect on their own learning. Exploratory tasks demand students to engage actively in the construction of their knowledge by solving situations where there is no clear solution or method. Sometimes, they are also challenged to ask questions or extend the purpose of the task. Students need to interpret the given information, develop strategies, represent and communicate their solutions. This promotes the understanding of representations, concepts, and procedures, and also develops the ability to argue about ideas, as they communicate such ideas to others. Work on exploratory tasks develops usually in three phases (Ponte, 2005): (i) presenting and interpreting the task; (ii) carrying out the task individually, in pairs, or in small groups; and (iii) presenting and discussing results and final synthesis.

Geometric reasoning

The study of geometry is the natural context to develop and use visualization, special reasoning and geometric modeling to solve problems (NCTM, 2000). Despite the growing focus on geometric reasoning and visualization in research, clarification of their meanings is still missing (Gutiérrez, 1996). This is even more complicated by the many expressions used with similar meanings (geometric reasoning, visual reasoning, visualization, spatial thinking…). For example, for Battista (2007) "geometric reasoning consists, first and foremost, of the invention and use of formal conceptual systems to investigate shape and space" (p. 843), a definition we may find too broad. Also, the van Hiele's model describes how students' geometric reasoning develops and includes five levels: 1) visual-holistic reasoning; 2) descriptive-analytic reasoning; 3) relational-inferential reasoning; 4) formal deductive proof; and 5) rigor (Battista, 2009). These levels cover different forms of reasoning. So the need to investigate the development of geometric reasoning drove us to ask what is specific of this kind of reasoning and what main features does it have. A possible approach to study geometric reasoning consists in analyzing it from its processes, which are present in other areas but have some specificity in geometry. In this paper we will focus on the processes of defining and classifying.

Defining is a crucial activity in mathematics. For de Villiers, Govender and Patterson (2009), it is so import as solving problems, conjecturing or proving and, despite that, is much neglected in mathematics teaching. Their work with students in grades 9 to 12 suggests that producing definitions improves students' understanding of geometric definitions and concepts. Zazkis and Leikin (2008) emphasize that, for teachers to be able to support students in this process, they need to be competent in performing it. In a study involving prospective teachers, the construction and analysis of definitions for square showed their ability to distinguish necessary and sufficient conditions, use adequate language and show conceptions about defining.

The process of defining implies also classifying because of their mutual relationship:

> The classifications of any set of concepts implicitly or explicitly involves defining the concepts involved, whereas defining concepts in a certain way automatically evolves their classification. (de Villiers et al., 2009, p. 191)

In the perspective of Mariotti and Fischbein (1997), the process of defining must also be considered as a component of geometric reasoning. For those investigators,

> a classification task consists of stating an equivalence among similar but figurally different objects, towards a generalization. That means overcoming the particular case and consider this particular case as an instance of a general class. In other terms, the process of classification consists of identifying pertinent common properties, which determine a category. (pp. 243-244)

In a study with grade 6 students, those investigators found that classifications often resort to structural criteria which are not immediately clear and very often conflict with perceptual criteria we are used to refer in spontaneous classifications. Hence, achieving correct definitions makes students to question their prototypes which frequently introduces properties perceptually relevant that do not conform to the general requirements of the definition.

METHODOLOGY

This paper addresses an investigation with an intervention, in order to change practices and enhance teachers' preparation in geometry. The research focus is on learning in context, starting from the conception of strategies and teaching tools, following a design-based research as methodology, in the form of a prospective teacher experiment (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003) in which the teacher also plays the role of researcher. We expect to run trough cycles of creation and revision, trying to deal with the problems that we will find along the way. At the present time, a first cycle was conducted involving 60 prospective teachers. The participants are in the second year of their teacher education program and attend a curricular unit of geometry taught by the first author of this paper. The study of quadrilaterals was developed in five lessons, following three sequential steps suggested by de Villiers et al. (2009): (i) to investigate the properties of quadrilaterals using the dynamic geometry environment (DGE) Geogebra; (ii) to classify them; and (iii) to define quadrilaterals. In the first step, they solved the clown task (Battista, 2007) adapted for Geogebra, where one has to manipulate quadrilaterals to overlap others, forcing them to use the relations among them (e.g., a rectangle may overlap a square but not the opposite). Afterwards, the participants registered all the properties that they found in each quadrilateral. In the second step, they classified the figures using a flowchart and a Venn diagram with the purpose of realizing that different criteria lead to different organizations. In addition, the participants also worked on a definition task.

Data gathered includes the participants' records of tasks solved in the classroom, an assessment task and reflections concerning quadrilaterals collected from portfolios. We also present the results of two multiple choice questions about quadrilaterals addressed in an initial individual diagnostic test. In the first question the participants identified relations among quadrilaterals, and in the second one they had to decide on possible definitions for square. The data was analyzed through several processes. Regarding the process of defining we adopted the categorization of de Villiers' et al. (2009): economical definitions, correct definitions and incorrect definitions. In this last case we considered definitions containing necessary properties but insufficient to define the intended set; in this category are also the definitions presenting properties that do not apply to some or all objects. Correct definitions present properties necessary and sufficient; if those properties are minimal, the definition is economical. In respect to the processes of classifying, the categorization emerged from the data, and we refer the comprehension of inclusive classification of quadrilaterals and the use of logical reasoning and communication skills.

RESULTS

In the first lesson, 57 prospective teachers solved the diagnostic test. The results show that only 25% considered that all squares are rectangles (but not the opposite) and 39% considered incorrectly that all quadrilaterals with two pairs of congruent sides are rectangles. Confronted with four possible definitions for square, 86% chose correctly "Polygon with four congruent sides and four congruent angles", but 75% also pointed "Polygon with four congruent sides". Only 23% considered valid "Quadrilateral with congruent and perpendicular diagonals". These results are not significantly different for participants that studied and did not studied mathematics in high school. They show that most of the participants ignored the relations between quadrilaterals and did not notice properties related to diagonals or lines of symmetry. In addition they were very connected to rigid prototypes and reasoned about figures by comparison to those prototypes, which is associated to van Hiele's level 1. Also they seemed to accept that a correct description of the quadrilateral may function as a definition.

The first two tasks of the sequence confronted the participants with their previous conceptions and made them realize there are relations among figures they did not know or expect and helped them to understand these relations:

> When I began to solve this task, I thought I would only recall some ideas about quadrilaterals. However, trough out the activity not only I recall them but also I was able to fit in to my head the hierarchy between some quadrilaterals. (Reflection written in the participant's portfolio about the classification task)

In the definition task, the participants worked in small groups, registered their answers which were discussed collectively at the end of the lesson. They were asked to: 1) Identify all the rectangles' properties; 2) Propose two different definitions for rectangle; 3) Propose two different definitions for parallelogram.

In respect to 1), most of the groups identified correctly all the main properties of rectangles (using sides, angles, diagonals and symmetry). Questions 2) and 3) show that they understood that there is no need to present all properties of an object to define it and most produced correct definitions, which is associated to van Hiele's level 2 (Battista, 2009). The next response is an example of a correct definition for rectangle, in which one of the properties is valid but unnecessary:

> Group A: Rectangles' properties: 4 right angles; 2 by 2 parallel sides; 2 lines of symmetry; bisected diagonals; congruent diagonals.
>
> Definition: quadrilateral with 4 right angles and 2 lines of symmetry.

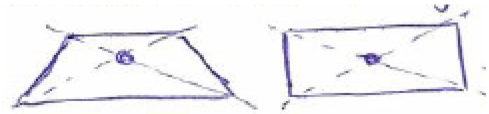
Although less frequent, some definitions were incorrect:

> Group A: Parallelogram: quadrilateral without lines of symmetry.
>
> Group B: A parallelogram is a figure composed by 2 paires of congruent and parallel sides, forming 2 acute angles (opposite) and another 2 obtuse (opposite).
>
> Group E: Rectangle: The diagonals intercept in the center but are not perpendicular;
>
> 2 symmetry lines (1 horizontal, 1 vertical) passing in the center of the figure.
>
> Group F: Rectangle: Geometric figure with 4 sides where the length should be bigger than the height.
>
> Parallelogram: Geometric figure similar to rectangle, where the shorter lines are oblique.

The definitions for parallelogram proposed by groups A and B exclude all rhombuses in the first case and all the rectangles in the second, so their definitions are not inclusive. Similarly, the first definition presented by group E excludes squares. These examples show some difficulty to abandon previous conceptions and recognize the hierarchical organization of quadrilaterals. Still in group E, the second definition is incorrect because it does not exclude some rhombuses. Yet, the more striking feature of this definition is that it is dependent of the position that rectangles are usually presented. Group F's response is the only one that considers as properties the relations between the dimension of the sides and its position. Although incorrect, these definitions were presented collectively, which led into an important discussion. Some students argued about their validity giving counter-examples or correcting the statements and others noticed and reflected on their own misunderstandings.

Finally, some examples of economical definitions demonstrate an interesting analysis, where students used less usual properties they discovered with Geogebra:

Group C: Rectangle: Quadrilateral with two congruent and bisecting diagonals.



Parallelogram: Each diagonal divides it into congruent triangles.

Group D: Rectangle: Quadrilateral with 2 lines of symmetry passing through the middle points of opposite sides.

In the first definition of group C, the prospective teachers draw a quadrilateral where the diagonals do not bisect so they justify the need to include this property. The second, although roughly written, is very interesting because the word "each" makes a difference (one diagonal would not be enough because of kites). Group D presents a definition focused on the lines of symmetry, but stating their position which is necessary (all rhombuses have also two lines of symmetry in a different location).

Overall, we found four types of problems. Producing economical definitions was the most common difficulty and the hardest to overcome, especially because the participants did not know how to be sure that the properties were sufficient to identify each quadrilateral. A second problem that came up some times was the production of non-inclusive definitions. Even for participants that seemed to understand previously the hierarchical relation between quadrilaterals, sometimes they stopped to consider it, showing difficulties to let go previous conceptions. All these cases correspond to van Hiele's level 2, according do Battista (2009). The third problem, happened in very few cases and corresponds to definitions linked to certain positions or relations between parts of the quadrilaterals, clearly associated to frequent prototypes (corresponding to van Hiele's level 1). Despite their low frequency, these cases must keep us aware of how striking the systematic exposure to rigid prototypes may be (Yu, Barret, & Presmeg, 2009). Finally, there was only one definition containing an insufficient property to define the quadrilateral.

The previous examples demonstrate some difficulties, but also some interesting successes if we remember that it was the first time that these participants defined something. To formulate definitions implies to investigate invariants. We must identify the common properties to all the elements we include in that class, mobilizing inductive reasoning and visual abilities, in particular visual discrimination and perceptual constancy (Gutierrez, 1996). So, given the fact that most of the definitions were correct, we consider that as a positive indicator regarding those abilities and inductive reasoning. The few participants that produced economical definitions moved to van Hiele's level 3 (Battista, 2009) and showed a significant improvement. Given the fact that formulating economical definitions involves also deductive reasoning, it appears the participants showed more difficulty in it.

The production of definitions was a good opportunity for the participants to learn about the quadrilaterals and to revise their conceptions about the process of defining, as this reflection shows:

> This task raised some doubts because, before we done it, I thought I knew the definitions of each figure, I thought there existed only one for each figure . . . I came across basic definitions about square or rectangle completely different from what I learned until then. To define figures I never had use angles, diagonals or even lines of symmetry; indeed, I was unaware of their major role. (Reflection written in the participant's portfolio)

Regarding the process of classifying, the work we have developed prompted most of the participants to consider quadrilaterals as classes of figures. However, this evolution does not happen all at once. It is possible that an individual recognize some relations and others do not. The following response regards a question in the final test, where the participants were asked to comment on two sentences: *All kites are squares*; *there are trapeziums with perpendicular diagonals*.

> The first sentence is wrong. Kites are not squares. The squares can be kites because they have two equal consecutive sides.
>
> The second sentence is wrong. A quadrilateral with perpendicular diagonals is a rhombus, which doesn't belong to trapezium's family.

These answers show several difficulties we also identified in other cases. In the first place, this participant recognized that a square is a kite, but did not recognize that a rhombus is a trapezium, supporting the conclusion that learning to classify is progressive and is not independent of the objects it regards. Second, it shows a logical problem: for the second sentence to be true, it is not necessary that all the trapeziums have perpendicular diagonals, so one counterexample does not deny that statement. Third, a problem of rigor in communicating: instead of "Kites are not squares", one should say "Some kites are not squares" and also the word "pairs" is missing from the kite's description. Communicating using the precise words has a fundamental role in the processes we are dealing with. A prospective teacher asked once during a lesson: "If a parallelogram is a trapezium, why do they have different names?", a question that shows difficulties in interpretation.

CONCLUSION

In the beginning of the experience, the prospective teachers showed weak knowledge about quadrilaterals and their relations. However, the work on the sequence of tasks (investigating quadrilaterals' properties, classifying and defining) seems to have promoted their reasoning and the reconstruction of their knowledge. In the definition task, most of the participants understood the meaning of the process itself and presented mostly correct definitions using properties that they previously ignored, showing the comprehension of the underlying concepts, which supports de Villiers' et al. conjecture (2009). However, the participants produced economical definitions in few cases, suggesting that they perform better in inductive rather than deductive reasoning. Classifications (associated or not with definitions) showed, in some cases, a conflict between prior classifications, based on perception, and structural criteria that rules geometrical classifications, which is fundamental to the learning process (also a result indicated by Mariotti and Fischbein, 1997). The process of classifying also mobilized logical reasoning and communication, which presented difficulties for some participants. However, the nature of the work developed in classes favored discussion and negotiation of meanings, which is essential to overcome those difficulties (de Villiers, 1994). This idea lead us to conclude that the exploratory work in which the participants engaged, using a DGE, allowed them to investigate and discuss their findings and construct their knowledge in a meaningful way. As the testimony of a prospective teacher shows, reflection may play an important role in becoming aware of personal preconceptions and knowledge, which is an essential part of teacher education (Ponte & Chapman, 2008).

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